Problem Set # 1

1. (a) Relevant parameters:
   - R: radius
   - t: time
   - E: energy
   - ρ: density of the ambient air

   Corresponding units:
   - $[R] = m$
   - $[t] = s$
   - $[E] = kg\frac{m^2}{s^2}$
   - $[\rho] = \frac{kg}{m^3}$

   Buckingham's Π- theorem:
   #non-dim. groups = #parameters − #units =4 − 3 = 1

   (b) $\Rightarrow \Pi_1 = R^\alpha \cdot t^\beta \cdot E^\gamma \cdot \rho^\delta$
   - $[s]: \ 0 = \beta - 2\gamma$
   - $[kg]: \ 0 = \gamma + \delta$
   - $[m]: \ 0 = \alpha + 2\gamma - 3\delta$

   $\Rightarrow$ 3 equations but 4 unknowns

   <u>Choose</u>: $\alpha = 1$
   - $\gamma = -\delta$
   - $0 = 1 - 2\delta - 3\delta \Rightarrow \delta = \frac{1}{5} \Rightarrow \gamma = -\frac{1}{5}$
   - $\beta = 2\gamma = -\frac{2}{5}$

   $\Rightarrow \Pi_1 = R \cdot t^{-\frac{2}{5}} \cdot E^{-\frac{1}{5}} \cdot \rho^{\frac{1}{5}} = R \cdot \left(\frac{\rho}{Et^2}\right)^{\frac{1}{5}}$
   $\Rightarrow E = \frac{R^5\rho}{t^2} \cdot \Pi_1^{-5}$

   (c) What is Π_1?

   $\Rightarrow$ Physics enter: From gas dynamics of shock waves (or experiments) $\rightarrow \Pi_1 \approx 1$
   $E = \frac{R^5\rho}{t^2} \qquad \rho \approx 1.2\frac{kg}{m^3}$

   $E = \frac{(59m)^5 \cdot 1.2\frac{kg}{m^3}}{(3.26 \cdot 10^{-3}s)^2} = 8,1 \cdot 10^{13}J = 8.1 \cdot 10^{10}kJ$

   (d) $E = 8.1 \cdot 10^{10}kJ \cdot \frac{1g}{4kJ} = 20.3$ kilotons

   (e) Homework (voluntary)
   i. What happens if you consider the ambient pressure to be another relevant parameter in the above problem?

ii. Use the Buckingham's Π-Theorem to determine the non-dimensional group(s) describing the motion of a linear, free (no damping) oscillator. What is the value of the non-dimensional group?

In summary Buckingham's Π-Theorem is a very powerful tool, but the result, especially when used without the underlying equations,strongly depends on the correctness of the guessed parameters. The solution is always only correct up to a constant, whose value needs to be determined from "real" physics.

2. **Isotropic Decaying Turbulence**

- Prerequisites

In DNS (Direct Numerical Simulation) the complete Navier-Stokes-Equations are solved on a computational grid.
$\frac{\partial U_j}{\partial t} + U_i \frac{\partial U_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 U_j}{\partial x_i \partial x_i}$

In our case the turbulence will decay from an initial field, as there is no "stirring" mechanism (production term) in the solved equations, so that viscosity will gradually damp the fluctuations.

Some statistical notions (will be introduced in detail in the next lecture):

Homogeneity: statistical quantities are invariant to a translation in space
Isotropy: statistical quantities are invariant to a rotation in space

Mean: $\langle U \rangle$, $\langle V \rangle$, $\langle W \rangle$

Rules for mean

- $\langle \langle a \rangle \rangle = \langle a \rangle$
- $\langle a \langle b \rangle \rangle = \langle a \rangle \langle b \rangle$

Variance (measure for fluctuations around the mean):

$$\left\langle (U - \langle U \rangle)^2 \right\rangle = \left\langle U^2 - 2U \langle U \rangle + \langle U \rangle^2 \right\rangle = \left\langle U^2 \right\rangle - 2 \langle U \rangle^2 + \langle U \rangle^2 = \left\langle U^2 \right\rangle - \langle U \rangle^2 \tag{1}$$

For isotropic flow: $\Rightarrow \left\langle U^2 \right\rangle = \left\langle V^2 \right\rangle = \left\langle W^2 \right\rangle$

- Solution

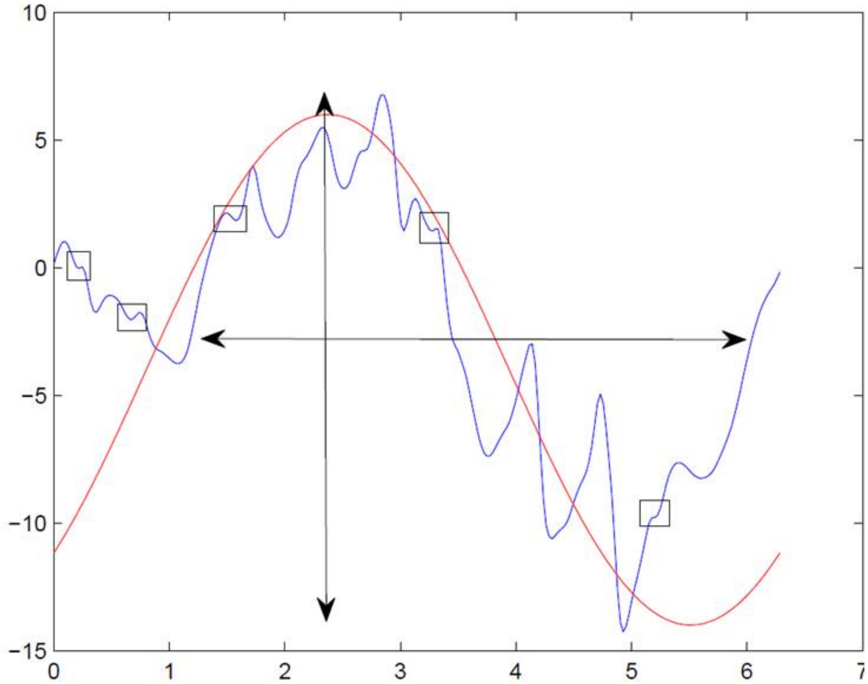

Figure 1: U-component of the velocity vector along the line $j = k = 10$.

(a) Figure 1 shows the velocity profile along a line. By looking at it, we can visually estimate the small and large scales. The red curve is a sinusoidal wave given by

$$U(x) = 10\sin(x - 0.8) - 4\,.$$

This sinusoidal wave more or less represents the largest fluctuations. The magnitude of this wave is about $u^{'} \approx 10$ or $u^{'} \approx 20$. Its length scale is about $L \approx 3$ or $L \approx 6$.
The small fluctuations are enclosed in the boxes. The magnitude of the velocity fluctuations is much smaller, of order $u^{'} \approx 0.1$. Their corresponding length scales are of order $l \approx 0.1$.
It is interesting to note that the large velocity fluctuations correspond to large length scales and small velocity fluctuations to small length scales.

(b) If the Reynolds number were increased, the large scales would be virtually unchanged while the smaller scales would become even smaller.

(c) The mean and variance of the different velocity components can be evaluated with the attached MATLAB code. The results would be

$$\begin{aligned} <U> &= -6.8328 \cdot 10^{-12} \quad , \quad <u^2> = 16.3888 \\ <V> &= -3.7326 \cdot 10^{-11} \quad , \quad <v^2> = 16.9161 \\ <W> &= 3.9605 \cdot 10^{-12} \quad , \quad <w^2> = 15.4504 \end{aligned}$$

The means of the three components over the entire 3D field are zero within machine accuracy. The variances of the three velocity components are roughly the same since the turbulence is isotropic.

Possible reasons for deviations from perfect isotropy:

– not fully statistically converged
– domain not spherical

```
%PS1_IsotropicTurbulence.m

close all;
clear all;

%Read the DNS data file.
[u,v,w]=read_box;

%Get the size of the datafile
[nx,ny,nz] = size(u);

%Create a space vector.
dx = 2*pi/256;
x = (0:255) * dx;

%Plot along the line j=k=10.
plot(x,u(:,10,10));

%Compute the mean of the the three velocity components.
umean = sum(sum(sum(u)))/(nx*ny*nz);
vmean = sum(sum(sum(v)))/(nx*ny*nz);
wmean = sum(sum(sum(w)))/(nx*ny*nz);

%Compute the variance of the three velocity components.
uvar = sum(sum(sum((u-umean).^2)))/(nx*ny*nz);
vvar = sum(sum(sum((v-vmean).^2)))/(nx*ny*nz);
wvar = sum(sum(sum((w-wmean).^2)))/(nx*ny*nz);
```

Figure 2: Possible Matlab Code